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Small-scale intermittency of turbulent flows§

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Abstract. The correction to the 1941 Kolmogorov theory is estimated using the theory of stochastic differential equations. Results are in good agreement with experimental data.

1. Introduction

Many attempts have been made during the last few years to explain the deviation from the 1941 Kolmogorov theory (K41). Landau and Lifshitz were the first to point out this deviation (see Landau and Lifshitz 1971, p 158). Among others, Gurvich and Yaglom (1967), Novikov and Stewart (1964), Obukhov (1962) and Kolmogorov (1962) then tried to formulate a theoretical explanation for this phenomenon and to evaluate the deviation experimentally. It is now generally accepted that the data on energy spectra in the inertial range in three-dimensional flows are compatible with the following expression:

$$E(K) = C\langle\epsilon\rangle^{2/3}K^{-5/3}(K/K_0)^{-B} \tag{1}$$

where $B \simeq 0{\cdot}1 \div 0{\cdot}2$ experimentally (Monin and Yaglom 1975). Frisch *et al* (1978) have related B to the Hausdorff dimension D of intermittent flows (anomalous dimension) as introduced by Mandelbrot (1974):

$$B = \tfrac{1}{3}(3-D) = \tfrac{1}{3}\mu, \qquad \mu = 3-D. \tag{2}$$

In this paper an estimate of μ is given using the theory of stochastic differential equations (Ventzel and Freidlin 1970). We believe that the idea by which this result is obtained may be a useful starting point for a new approach to understanding turbulence.

2. The model of intermittency

The following analysis is in the spirit of the work of Frisch *et al* (1978). A flow configuration in which vorticity is strongly localised is called 'intermittent'. Vortex tubes and sheets (Townsend 1951, Saffman 1968) are intermittent configurations, but there are probably more complicated situations in real turbulent flows (Frisch *et al* 1978). Vortex tubes and sheets actually have D equal to one and two, respectively, whereas experimentally D is greater than two. We introduce a hierarchy of scales $\{l_n\}$

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($l_n = b^{-n} l_0$, b being a number greater than one and l_0 the external length) in which intermittent configurations are present at random times. Let V_n^i be the characteristic intermittent velocity and P_n the probability of having an intermittent configuration at scale l_n. The mean value of any function f of the velocity field is given by

$$\langle f(V_n) \rangle = P_n f(V_n^i) + (1 - P_n) f(V_n^l) \tag{3}$$

where V_n^l is the corresponding non-intermittent velocity at the scale l_n. Here V_n is the velocity difference across a distance l_n. It follows that $V_n^l \ll V_n^i$ and therefore equation (3) may be written in the form

$$\langle f(V_n) \rangle \simeq P_n f(V_n^i). \tag{4}$$

The kinetic energy per unit mass in the scale l_n is given by

$$E_n = \int_{K_n}^{K_{n+1}} E(K)\, \mathrm{d}K \tag{5}$$

where $K_n = l_n^{-1}$. From the definition of V_n^i and making use of equation (4) we obtain

$$E_n \simeq P_n (V_n^i)^2. \tag{6}$$

$E(K_n)$ can therefore be written as

$$E(K_n) \simeq P_n (V_n^i)^2 l_n. \tag{7}$$

According to K41 theory the energy transfer ϵ_n from the l_n to the l_{n+1} scale is, on the average, independent of n and of viscosity. From dimensional considerations we therefore have

$$\langle \epsilon_n \rangle = \langle \epsilon \rangle \simeq P_n (V_n^i)^3 / l_n. \tag{8}$$

From equations (7) and (8) we obtain

$$E(K_n) \simeq \langle \epsilon \rangle^{2/3} K_n^{-5/3} P_n^{1/3}. \tag{9}$$

Note that if P_n is independent of K_n there is no deviation from K41 theory. To estimate P_n, the theory of differential stochastic equations is used, i.e. a random force acting on scales l_n is considered. This stochastic perturbation may be due to thermodynamic fluctuations (Ruelle (1979) has recently estimated this effect and has argued that it must be relevant for small scales) or to an external noise due to macroscopical perturbation (Forster *et al* 1977). It is possible to show in the theory of stochastic differential equations (Ventzel and Freidlin 1970) that the solution of the equation

$$\mathrm{d}X_i = b_i(\{X_j\})\, \mathrm{d}t + \sigma_{ij}\, \mathrm{d}W_j(t), \tag{10}$$

where $\mathrm{d}W_j(t)$ is the standard Wiener process, is equivalent to a Markov chain whose states are the stationary points (fixed points and limit cycles) of the deterministic equation

$$\mathrm{d}X_i/\mathrm{d}t = b_i(\{X_j\}). \tag{11}$$

Jumps between the states are at random times. For a physical approach to stochastic differential equations the reader is referred to Chandrasekhar (1943), while for a detailed mathematical theory reference is made to Gihman and Skorohod (1972; see also Jona-Lasinio (1979) for the use of stochastic differential equations to compute functional integrals in quantum mechanics).

V_n is considered to be given by a stochastic differential equation of the kind (10). Following Forster *et al* (1977), it could be assumed that the noise is acting with equal intensity on all scales in the inertial range. This is consistent with the Obukhov hypothesis of Markovian diffusion in turbulent flows (see Monin and Yaglom 1975, p 571). However, this assumption will later be seen not to be a key one, because the mean times of exit can be estimated from stationary states using just heuristic arguments. The important point is the idea that the velocity field satisfies a stochastic differential equation of the kind (10) and therefore has the important property of 'jumping' between the stable and unstable stationary states of the system. There are assumed to be two kinds of stationary states:

(a) intermittent configurations with high vorticity (unstable);

(b) non-intermittent configurations (stable).

The assumption concerning the stability of non-intermittent stationary configurations must not be regarded as applying to the Navier–Stokes equations but to the equations for the evolution of V_n. Let l_n be a given scale of motion; V_n is the sum of all Fourier harmonics of the velocity field with wavenumber K satisfying the relation

$$K_n \leqslant K < K_{n+1}$$

where $K_n = l_n^{-1}$. Therefore V_n is a collective variable of the same kind as the one used by Desnyansky and Novikov (1974a, b). The words stability and instability in the assumptions (a) and (b) are thus to be taken in a statistical sense. This is consistent with the usual assumptions concerning turbulent flows.

The system of course spends more time in the non-intermittent configurations than in the intermittent ones. One may then think of the time evolution of any intermittent variable at a given scale l_n, for instance the velocity V_n, as a succession of rectangular pulses of mean length τ_n and amplitude V_n^i separated by a mean time interval T_n. (Note that this is the assumption of Novikov and Stewart 1964.) τ_n is the mean time of exit from intermittent configurations, while T_n is the mean time for a jump from a non-intermittent to an intermittent configuration ($T_n \gg \tau_n$).

From the ergodic properties of the processes we obtain

$$P_n = \lim_{T\to\infty} \frac{(\Delta T)_n}{T} \tag{12}$$

where $(\Delta T)_n/T$ is the fraction of time the system spends in the intermittent configurations. Clearly

$$P_n \simeq \tau_n/T_n. \tag{13}$$

τ_n and T_n in the inertial range must depend only on $\langle\epsilon\rangle$, l_n and large scales properties (like l_0). From this assumption we immediately obtain

$$\tau_n \sim \langle\epsilon\rangle^{-1/3} l_n^{2/3} = \langle\epsilon\rangle^{-1/3} K_n^{-2/3}. \tag{14}$$

Once there is a jump to an intermittent configuration, all the scales of the flow exhibit high values of vorticity; this implies that T_n is independent of n:

$$T_n \sim \langle\epsilon\rangle^{-1/3} K_0^{-2/3}. \tag{15}$$

From equations (13), (14) and (15) we have

$$P_n \sim (K_n/K_0)^{-2/3}. \tag{16}$$

Equation (9) therefore becomes

$$E(K_n) \sim \langle \epsilon \rangle^{2/3} K_n^{-5/3} (K_n/K_0)^{-2/9} \tag{17}$$

i.e. $\mu = \frac{2}{3}$ which is in good agreement with the known experimental values.

3. Higher-order moments

Using (16), an estimate may be given of the dimensionless structure functions

$$a_p(l) = \langle \delta V(l)^p \rangle / \langle \delta V(l)^2 \rangle^{p/2}$$

where $\delta V(l)$ is any fluctuating component of the difference between the velocities of two points $\boldsymbol{r}$ and $\boldsymbol{r}'$ with $|\boldsymbol{r} - \boldsymbol{r}'| = l$. From equations (4), (8) and (15) we obtain

$$a_p(l) \sim (l/l_0)^{\xi_p} \tag{18}$$

where

$$\xi_p = \tfrac{1}{3}(2-p). \tag{19}$$

The deviation of the small-scale probability distribution from the Gaussian distribution is given quantitatively by the skewness S and the kurtosis F defined as follows:

$$S = -\left\langle \left(\frac{\partial u}{\partial x}\right)^3 \right\rangle \Big/ \left\langle \left(\frac{\partial u}{\partial x}\right)^2 \right\rangle^{3/2}, \qquad F = \left\langle \left(\frac{\partial u}{\partial x}\right)^4 \right\rangle \Big/ \left\langle \left(\frac{\partial u}{\partial x}\right)^2 \right\rangle^2. \tag{20}$$

It is well known that the largest contributions to the velocity gradient are in the dissipative scale l_d (defined below), so that we obtain

$$S \simeq -a_3(l_\mathrm{d}), \qquad F \simeq a_4(l_\mathrm{d}). \tag{21}$$

l_d is defined as that scale for which the dissipative time $\tau^\mathrm{d} \sim l_\mathrm{d}^2/\nu$ is of the same order of magnitude as the inertial time $\tilde{\tau}$:

$$\tilde{\tau} \sim E(l_\mathrm{d}^{-1})/l_\mathrm{d}\langle \epsilon \rangle.$$

After some algebra we obtain

$$l_\mathrm{d} \sim l_0 R^{-9/10}, \qquad R = \langle \epsilon \rangle^{1/3} l_0^{4/3} \nu^{-1}. \tag{22}$$

From equations (22), (21), (19) and (18) we obtain

$$S \sim R^{3/10}, \qquad F \sim R^{3/5}. \tag{23}$$

The experimental value of F gives (Van Atta and Chen 1970, Van Atta and Park 1972)

$$F \sim R^\alpha, \qquad \alpha \simeq 0{\cdot}6, \tag{24}$$

in agreement with (23).

4. The case of two-dimensional turbulence

The argument given for three-dimensional turbulence can be repeated for two dimensions. Because of vorticity conservation there is an enstrophy cascade to small scales and an inverse energy cascade to large scales (Kraichnan 1967, Batchelor 1969). If the energy input is on scale l_0, then it is commonly assumed (Kraichnan 1967, 1975, Batchelor 1969) that there are two different inertial ranges. The first one, $l \ll l_0$, is dominated by the enstrophy cascade and has a power spectrum (Rose and Sulem 1978)

$$E_1(K) \sim \langle\eta\rangle^{2/3} K^{-3}[\ln(K/K_0)]^{-1/3} \tag{25}$$

where η is the enstrophy dissipation. The second inertial range, $l \gg l_0$, has an inverse energy cascade and a power spectrum

$$E_2(K) \sim \langle\epsilon\rangle^{2/3} K^{-5/3}. \tag{26}$$

Our approach may be used to compute the corrections due to intermittency for equations (25) and (26). In the first inertial range τ_n is a function of $\langle\eta\rangle$ and l_n. Since $\langle\eta\rangle$ has the dimensions of a time to the power minus three, τ_n is n-independent. Therefore no corrections to the scaling law (25) come from intermittent configurations. This is in agreement with theoretical considerations of Kraichnan (1975). On the other hand, in the second inertial range, equation (26), the three-dimensional argument given before may be followed straightforwardly, thus obtaining

$$E_2(K) \sim \langle\epsilon\rangle^{2/3} K^{-5/3}(K/K_0)^{2/9}. \tag{27}$$

Note that the exponent of the correction has changed sign as previously suggested by Kraichnan (1975).

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